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LETTER TO THE EDITOR

Log-concave functions and convex interaction terms for quantised fields

Jan Tarski

Fakultät für Physik, Universität Bielefeld, 48 Bielefeld 1, FDR

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Abstract. One considers those models of quantised fields for which the interaction Hamiltonians, with counter terms included, are convex functions of the fields. Some simple properties of these models (and of related ones) are obtained with the help of the theory of log-concave functions. In particular, the form of ground-state functionals is specified.

1. Introduction

In two recent articles Brascamp and Lieb (1975, 1976) developed various properties of log-concave functions. They discussed applications in particular to the one-dimensional Coulomb plasma and to the Ising model. In the present letter we should like to indicate the usefulness of these functions for the study of certain models of quantum field theory.

The models in question are those for which the interaction Hamiltonian H_{int} is a convex function (or functional) of the fields. This means that H_{int} without counter terms, or renormalisation terms, must be convex, and that the counter terms must not upset the convexity. (We consider only boson fields without derivative couplings, and then the fields at a fixed time can be regarded as numerical entities.)

Aside from free fields, there are three classes of models which satisfy the foregoing criterion of convexity.

(i) The Høegh-Krohn model, where the coupling is constructed as a superposition of exponentials in two dimensions (cf Høegh-Krohn 1971, Simon 1974 for further details). In this model the counter terms combine to yield an overall multiplicative effect. In fact, for a single exponential one has, formally

$$:\mathrm{e}^{c\phi}: \sim Z\mathrm{e}^{c\phi}, \qquad Z \sim \exp(-\tfrac{1}{2}c^2\langle\phi(0)^2\rangle_0). \tag{1}$$

(For a field with ultraviolet cut-off, $Z_\kappa > 0$ and $\mathrm{e}^{c\phi_\kappa}$ is convex, hence $:\mathrm{e}^{c\phi_\kappa}:$ is also convex, and this property will survive as $\kappa \to \infty$.)

(ii) The quadratic interaction in any number of dimensions (cf Ginibre and Velo 1970, Rosen 1972).

(iii) Models with ultraviolet cut-offs and without counter terms, with exponential or convex polynomials or other convex couplings.

The convexity property remains valid if we introduce non-negative functions which yield spatial cut-offs. We assume, besides convexity, that H_{int} is an even function of ϕ, and that the free Hamiltonian $H^{(0)}$ has a non-zero mass. Some additional assumptions about the models will be introduced later in this letter.

For the models with a spatial cut-off we will deduce the form of equation (9) given later for the ground-state functionals. Moreover, we discuss briefly the characteristic functionals of fixed-time measures, the analyticity of these functionals, and the removal of spatial cut-off.

In the text we will refer also to models defined by total Hamiltonians of the following kind (cf Coester and Haag 1960, Tarski 1969, 1972a):

$$H_{\text{total}} = \frac{1}{2}\int d^3u \left(\dot{\phi}(u) + i\frac{\delta\Lambda(\phi)}{\delta\phi(u)}\right)\left(\dot{\phi}(u) - i\frac{\delta\Lambda(\phi)}{\delta\phi(u)}\right) + \text{CT}, \tag{2a}$$

$$\Lambda = \Lambda^{(0)} + \Lambda_1 = \Lambda^{(0)} + \int d^3u[(\phi * h)(u)]^{2n} g(u), \tag{2b}$$

$$\Lambda^{(0)} = \frac{1}{2(2\pi)^{3/2}}\int d^3u_1\, d^3u_2\, \lambda^{(0)}(u_1 - u_2)\phi(u_1)\phi(u_2), \tag{2c}$$

and $\lambda^{(0)}$ has the Fourier transform $\tilde{\lambda}^{(0)}(p) = (p^2 + m^2)^{1/2}$. We will refer to these models as Λ-models. The counter terms are those associated with the Wick ordering of $H^{(0)}$, and the functions $g, h \in \mathscr{S}$ $(g \geqslant 0)$ provide a spatial and an ultraviolet cut-off for interactions, respectively.

Here H_{int} is not in general a convex function of ϕ. However, for the ground-state functional we have the expression

$$\Psi_0(\xi) = (\text{constant}) \exp(-\Lambda^{(0)}(\xi) - \Lambda_1(\xi)), \tag{3}$$

and so Ψ_0 has already the form of equation (9). For the Λ-models, the properties of log-concave functions allow a simplified derivation of the basic inequalities.

2. Summary concerning log-concave functions

These are the functions of the form $e^{-K(x)}$ where K is convex (so that $-K$ is concave). For example, let $K(x) = \langle x, Ax\rangle$ with $A \geqslant 0$. The possibility $K = +\infty$, $e^{-K} = 0$, is admitted. The following properties of log-concave functions, defined for $x \in R^n$, are basic. For the proofs of (*a*) and (*b*), see Brascamp and Lieb (1975, 1976). All integrals that we write are assumed to exist.

(*a*) The product and the convolution of log-concave functions are log-concave. Moreover, if W is Gaussian on R^{n+m} and if we set

$$\begin{aligned} I_2(x)\int d^my\, W(x,y) &= \int d^my\, WF \\ I_1(x) &= \int d^my\, F(x,y), \end{aligned} \tag{4}$$

then (F is log-concave on $R^{n+m} \Rightarrow$ each I_j is log-concave).

(*b*) Consider the following potential V on R^n, where K is convex, and where $\exp(-tV) \in L_1$ for all $t > 0$:

$$V(x) = \tfrac{1}{2}a^2x^2 + K(x), \qquad a \geqslant 0. \tag{5a}$$

Then the ground-state wavefunction ψ_0 for the operator $-\frac{1}{2}\nabla^2 + V$ has the following form, with F log-concave:

$$\psi_0(x) = \exp(-\tfrac{1}{2}ax^2)F(x). \tag{5b}$$

(*c*) We call any $g: R^n \to R^1$ even if $g(x) = g(-x)$, and we say that $f_+: R^1 \to R^1$ is even non-decreasing if in addition

$$f_+(y_1) \leqslant f_+(y_2) \qquad \text{when } |y_1| \leqslant |y_2|. \tag{6a}$$

Let W, $U: R^n \to R^1$, where W is fixed and U varies. We write, for $h: R^m \to R^1$,

$$\langle h \rangle_U = \int d^n x W(x) U(x) h(x^1, \ldots) \left(\int d^n x W(x) U(x) \right)^{-1}. \tag{6b}$$

Let $2f_\pm(y) = f(y) \pm f(-y)$. If W is Gaussian and G is even and log-concave, and $f: R^1 \to R^1$ is such that f_+ is even non-decreasing, then it is immediate (in view of (4); cf also lemma 1 of Tarski 1972a) that

$$\langle f_- \rangle_G = 0, \qquad \langle f_+ \rangle_G = \langle f \rangle_G \leqslant \langle f \rangle_1. \tag{7}$$

Note that an even log-concave function on R^1 is even non-increasing. (For even log-concave functions on R^n, this holds for the restriction to any line through the origin.)

(*d*) Assume G, W as before and $G = e^{-\lambda K}$. From (7) we obtain by differentiation

$$\langle fK \rangle_1 \geqslant \langle f \rangle_1 \langle K \rangle_1. \tag{8}$$

This is a Griffiths (1967) inequality obtained under an (apparently) new set of assumptions, which we recapitulate: the weight W Gaussian, K even convex, $f: R^1 \to R^1$ such that f_+ is even non-decreasing.

All these conclusions generalise directly to infinite-dimensional integrals. In particular, it is natural to employ generalised invariant measures (Tarski 1972b) in this context. Then in the case of Gaussian integrals, a Gaussian weight plays a role analogous to a finite-dimensional Gaussian (hence log-concave) factor. This circumstance was exploited in Brascamp and Lieb (1975, 1976) in the arguments leading to equation (5*b*).

3. The form of ground-state functionals

The analysis of the Λ-models has shown that detailed information about ground-state functionals can be quite useful. We therefore attempt to generalise (5) to quantised fields. These relations suggest that for convex H_{int} (cf equation (3)),

$$\Psi_0(\xi) = \exp(-\Lambda^{(0)}(\xi)) F(\xi), \qquad F \text{ log-concave}, \ \xi \in \mathscr{H}. \tag{9}$$

The Hilbert space $\mathscr{H}$ is real and is determined by $\Lambda^{(0)}(\xi)$ as the norm.

This functional can combine with a generalised invariant measure to yield an (ordinary) measure for computing equal-time expectations, in analogy to the case of ψ_0 of (5*b*) (cf Tarski 1969, 1972a). For the free-field case, $d\mu^{(0)}(\eta) \sim \mathscr{D}(\eta) \exp(-2\Lambda^{(0)}(\eta))$. (We do not write 'equal to', since the two objects are conceptually different.)

We can establish the form (9) under the following additional assumptions: (*a*) the cut-offs allow H_{int} to be represented as an operator on the Fock space, and $H^{(0)} + H_{\text{int}}$ is essentially self-adjoint; (*b*) the lowest eigenvalue E_0 of the self-adjoint extension H_{total} is isolated.

These assumptions are fulfilled for the Høegh-Krohn model. For quadratic interactions we may need an ultraviolet cut-off together with a spatial cut-off. However, for these interactions Ψ_0 is a Gaussian, i.e. it has approximately the form (3) with Λ_1 quadratic, and the removal of either cut-off can be readily controlled (see Ginibre and Velo 1970, Rosen 1972).

In view of (*a*), we can exploit the free Euclidean measure μ_E as in Simon (1974, chaps 3 and 5). In particular, Trotter's formula (e.g. Reed and Simon 1972) allows a justification of

$$\langle\Phi_1(\phi)\,e^{-tH_{\text{total}}}\Phi_2(\phi)\rangle_0=\int d\mu_E(\eta)\exp\left(-\int_0^t d\tau\, H_{\text{int}}(\eta_\tau)\right)\Phi_1(\eta_{\tau=t})\Phi_2(\eta_{\tau=0}). \qquad (10a)$$

One also sees as in Simon (1974, p 163ff) that E_0 is non-degenerate, and that the ground states $\Psi^{(0)}$ and Ψ_0 of $H^{(0)}$ and H_{total} respectively satisfy $\langle\Psi^{(0)},\Psi_0\rangle=a\neq 0$.

The proof of (5*b*) in Brascamp and Lieb (1976) makes use of the Green function for $\partial_t-\frac{1}{2}\nabla^2+V$. The analogous Green functionals in field theory are difficult to handle rigorously, and we avoid them through the following device. We decompose μ_E in a manner suggested by Tarski (1971, §§ 5 and 6) to obtain from (10*a*)

$$[\exp(-tH_{\text{total}})\Psi^{(0)}](\xi)$$

$$\underset{\text{df}}{=}\Omega_t(\xi)$$

$$=\int\mathscr{D}(\eta_1)\Psi^{(0)}(\eta_1)\int_{\substack{\eta_2(0)=\eta_1\\ \eta_2(t)=\xi}} d\mu_r(\eta_2)\exp\left(-\int_0^t d\tau H_{\text{int}}(\eta_2(\tau))\right) \qquad (10b)$$

The integral over η_2 yields an additional factor $\Psi^{(0)}(\eta_1)$, so that one recovers $d\mu^{(0)}(\eta_1)$. The measure μ_r is Gaussian, and the arguments of Brascamp and Lieb (1975, 1976) show that Ω_t has a form such as in (9). Let us decompose Ω_t as $a\,e^{-E_0t}\Psi_0+\Psi_{1,t}$ with $\langle\Psi_0,\Psi_{1,t}\rangle=0$ (cf above). Then $e^{tE_0}\Omega_t\to a\Psi_0$ as $t\to\infty$, and so Ψ_0 also has the form (9).

With regard to the smoothness of F in (9), we note here only that F is $\mu^{(0)}$-measurable. For a more detailed investigation into the smoothness and positivity of ground-state functionals, see Albeverio and Høegh-Krohn (1975).

In the foregoing we used the evenness of H_{int} only to justify easily the formulae involving functional integration for $e^{-tH_{\text{total}}}$. The form (9) is also valid for non-even (but convex) H_{int} which are bounded from below, but even this semi-boundedness is not necessary.

4. Characteristic functionals and their analyticity

The present section is based on the assumption of the form (9) for Ψ_0, with F even. (This is a consequence of evenness of H_{int}.) Let us suppose for the moment that $\Psi_0\in\mathscr{H}_{\Phi_{0\kappa}}$. Then we introduce the measure $d\mu\sim\mathscr{D}(\eta)|\Psi_0(\eta)|^2$ (continuous with respect to $d\mu^{(0)}$) and its characteristic functional, where $z\in C^1$,

$$C(z\alpha)=\int d\mu(\eta)\,e^{z\langle\eta,\alpha\rangle}=\int d\mu^{(0)}(\eta)F^2(\eta)\,e^{z\langle\eta,\alpha\rangle}. \qquad (11)$$

(Note that $\alpha\in\mathscr{H}$ but $z\alpha\in\mathscr{H}+i\mathscr{H}$.) By using (7) and by arguing as in Tarski (1969) we conclude that

$$|C(i\alpha)-C(i\beta)|\leq(2/\pi)^{1/2}\|B(\alpha-\beta)\|_2, \qquad (12)$$

where B is the (bounded) operator of multiplication in p space by $2^{-1/2}(p^2+m^2)^{-1/4}$.

The last inequality allows us to eliminate the spatial cut-off by selecting a suitable sequence of cut-off functions, and to obtain a Euclidean invariant theory, as in Tarski (1969). Then the measure ν associated with such a limiting theory is a weak (or 'narrow') limit of the measures μ, and in view of the discussion in Tarski (1972a, p 185), the following conclusions apply also to ν.

By comparing $C(z\alpha)$ with the corresponding quantity for the free theory and for z real (cf (7) again), we conclude that $C(z\alpha)$ is an entire function of z, of order at most 2. This analyticity leads directly to the assertion: *$C(\beta)$ is analytic when β varies over the space $\mathscr{H}+\mathrm{i}\mathscr{H}$.* To prove this, the following criterion for analyticity of a functional $S(\beta)$ defined on a complex Banach space is convenient (Field 1974): S is analytic if it is continuous and if its restriction to each complex line is analytic as a map $C^1 \to C^1$.

Thus the remaining task is to verify the continuity of C. We set $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, $\beta = \beta_1 + \mathrm{i}\beta_2$, and we use the estimate

$$|\mathrm{e}^{\langle\eta,\alpha\rangle} - \mathrm{e}^{\langle\eta,\beta\rangle}| \leq (|\langle\eta, \alpha_1 - \beta_1\rangle| + |\langle\eta, \alpha_2 - \beta_2\rangle|)(\mathrm{e}^{\langle\eta,\alpha_1\rangle} + \mathrm{e}^{\langle\eta,\beta_1\rangle}). \tag{13}$$

Schwartz' inequality and an argument such as the one leading to (12) now allow us to conclude that $|C(\alpha) - C(\beta)|$ can be majorised by an expression which is proportional to $\Sigma_j \|B(\alpha_j - \beta_j)\|_2$. Thus continuity and also analyticity follow.

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